Lecture 24

Limit Comparison Test

Suppose that $\sum a_n$ & $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c>0$, then either both series converge, or both diverge.

Proof: Since the terms $\frac{a_n}{b_n}$ are positive, and converge to c, there are numbers $m, M>0$ with $m<c<M$ and a number $N>0$ such that for $n \geq N$ we have $m < \frac{a_n}{b_n} < M$. Thus $mb_n < a_n < Mb_n$, and because all terms are positive:

$$m\sum_{n=N}^{\infty} b_n = \sum_{n=N}^{\infty} mb_n < \sum_{n=N}^{\infty} a_n < \sum_{n=N}^{\infty} Mb_n = M\sum_{n=N}^{\infty} b_n$$

$$\underline{m\sum_{n=N}^{\infty} b_n < \sum_{n=N}^{\infty} a_n}$$

1) $\sum a_n$ conv. $\Rightarrow \sum b_n$ conv

2) $\sum b_n$ div $\Rightarrow \sum a_n$ div

$$\underline{\sum_{n=N}^{\infty} a_n < M\sum_{n=N}^{\infty} b_n}$$

3) $\sum a_n$ div. $\Rightarrow \sum b_n$ div.

4) $\sum b_n$ conv. $\Rightarrow \sum a_n$ conv.

<u>1)+4)</u>

$\sum a_n$ conv. $\Leftrightarrow \sum b_n$ conv.

<u>2)+3)</u>

$\sum a_n$ div. $\Leftrightarrow \sum b_n$ div.

$\square$

Ex: Test the following series for convergence:

ⓐ $\sum_{n=1}^{\infty} \frac{1}{n^2-3}$ ⓑ $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$ ⓒ $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$

ⓓ $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^3 \pi^{-n}$ ⓔ $\sum_{n=1}^{\infty} \frac{1}{n!}$

Sol:

ⓐ $\frac{1}{n^2-3}$ looks like $\frac{1}{n^2}$, let $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n^2}$

$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2}{n^2-3} = 1$. So, since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, by the limit comparison test (LCT) $\sum_{n=1}^{\infty} \frac{1}{n^2-3}$ converges.

ⓑ $\frac{n^2-5n}{n^3+n+1}$, for very large n, behaves like $\frac{n^2}{n^3} = \frac{1}{n}$.

Let $\sum b_n = \sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^3-5n^2}{n^3+n+1} = 1$

So, since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\sum_{n=1}^{\infty} \frac{n^2-5n}{n^3+n+1}$, by the LCT.

ⓒ $\frac{\sqrt{n^4+1}}{n^3+n^2} > \frac{\sqrt{n^4}}{n^3+n^2} = \frac{n^2}{n^3+n^2} = \frac{1}{n+1} > \frac{1}{2n}$

Since $\sum_{n=1}^{\infty} \frac{1}{2n}$ diverges, by the comparison test, $\sum_{n=1}^{\infty} \frac{\sqrt{n^4+1}}{n^3+n^2}$ diverges too.

(d) $a_n = \left(1+\frac{1}{n}\right)^3 \pi^{-n} = \left(1+\frac{1}{n}\right)^3 \cdot \frac{1}{\pi^n}$

Since $\pi > 1$, π^n grows faster than n^p for any p. Thus we should expect convergence here because π^{-n} will dominate $\left(1+\frac{1}{n}\right)^3$. We know $\sum_{n=1}^{\infty} \pi^{-n}$ converges, so let $\sum b_n = \sum_{n=1}^{\infty} \pi^{-n}$.

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^3 = 1$$

So, by the LCT, $\sum_{n=1}^{\infty} \left(1+\frac{1}{n}\right)^3 \pi^{-n}$ converges.

(e)

$$\frac{1}{n!} = \frac{1}{1\cdot 2\cdot 3\cdot \cdots \cdot (n-1)\cdot n} = \frac{1}{1}\cdot\frac{1}{2}\cdot\frac{1}{3}\cdot \cdots \cdot \frac{1}{n-1}\cdot\frac{1}{n}$$

$$\leq 1\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot \cdots \cdot \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2^{n-1}}$$

So, $\sum_{n=1}^{\infty} \frac{1}{n!} \leq \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} < \infty$ since $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a convergent geometric series. Thus $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test.

(Fun fact: $\sum_{n=1}^{\infty} \frac{1}{n!} = e$. We'll show this later.)